$$\sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m$$ （$m = 1,2,\cdots$）の和の求め方について

m	$\sum_{k=1}^{n} k^m$	m	$\sum_{k=1}^{n} k^m$
1	$\frac{1}{2}n(n+1)$	6	$\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$
2	$\frac{1}{6}n(n+1)(2n+1)$	7	$\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$
3	$\frac{1}{4}n^2(n+1)^2$	8	$\frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)$
4	$\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	9	$\frac{1}{20}n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)$
5	$\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$	10	$\frac{1}{66}n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2 10n-5)$

（求め方）

1.　恒等式　$k(k+1)-(k-1)k = 2k$ に $k = 1,2,\cdots,n$ を代入して，辺々加えると

$$n(n+1) = 2\sum_{k=1}^{n} k \quad \therefore \sum_{k=1}^{n} k = \frac{1}{2}n(n+1) \quad \blacksquare$$

2.　2 つの恒等式　$k^2(k+1)-(k-1)^2 k = 3k^2 - k$，$k(k+1)^2-(k-1)k^2 = 3k^2+k$ を辺々加えると

$$k(k+1)(2k+1)-(k-1)k(2k-1) = 6k^2$$

これに $k = 1,2,\cdots,n$ を代入して，辺々加えると

$$n(n+1)(2n+1) = 6\sum_{k=1}^{n} k^2 \quad \therefore \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \quad \blacksquare$$

3.　恒等式　$k^2(k+1)^2-(k-1)^2k^2 = 4k^3$ に $k = 1,2,\cdots,n$ を代入して，辺々加えると

$$n^2(n+1)^2 = 4\sum_{k=1}^{n} k^3 \quad \therefore \sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2 \quad \blacksquare$$

4.　2 つの恒等式　$k^3(k+1)^2-(k-1)^3k^2 = 5k^4-2k^3+k^2$，$k^2(k+1)^3-(k-1)^2k^3 = 5k^4+2k^3+k^2$ を辺々加えると

$$k^2(k+1)^2(2k+1)-(k-1)^2k^2(2k-1) = 10k^4+2k^2$$

これに $k = 1,2,\cdots,n$ を代入して，辺々加えると　$n^2(n+1)^2(2n+1) = 10\sum_{k=1}^{n} k^4 + 2\sum_{k=1}^{n} k^2$

$$\therefore \sum_{k=1}^{n} k^4 = \frac{1}{10}\left\{n^2(n+1)^2(2n+1) - 2\times\frac{1}{6}n(n+1)(2n+1)\right\} = \frac{1}{30}n(n+1)(2n+1)\{3n(n+1)-1\} = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \quad \blacksquare$$

5.　恒等式　$k^3(k+1)^3-(k-1)^3k^3 = 6k^5+2k^3$ に $k = 1,2,\cdots,n$ を代入して，辺々加えると　$n^3(n+1)^3 = 6\sum_{k=1}^{n} k^5 + 2\sum_{k=1}^{n} k^3$

$$\therefore \sum_{k=1}^{n} k^5 = \frac{1}{6}\left\{n^3(n+1)^3 - 2\times\frac{1}{4}n^2(n+1)^2\right\} = \frac{1}{12}n^2(n+1)^2\{2n(n+1)-1\} = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1) \quad \blacksquare$$

6.　2 つの恒等式　$k^4(k+1)^3-(k-1)^4k^3=7k^6-3k^5+5k^4-k^3$，$k^3(k+1)^4-(k-1)^3k^4=7k^6+3k^5+5k^4+k^3$ を辺々加えると，$k^3(k+1)^3(2k+1)-(k-1)^3k^3(2k-1)=14k^6+10k^4$

これに $k=1,2,\cdots,n$ を代入して，辺々加えると

$$n^3(n+1)^3(2n+1)=14\sum_{k=1}^{n}k^6+10\sum_{k=1}^{n}k^4$$

$$\therefore \sum_{k=1}^{n}k^6=\frac{1}{14}\left\{n^3(n+1)^3(2n+1)-10\times\frac{1}{30}n(n+1)(2n+1)(3n^3+3n-1)\right\}=\frac{1}{42}n(n+1)(2n+1)\left\{3n^2(n+1)^2-(3n^3+3n-1)\right\}$$

$$=\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)\quad ■$$

7.　恒等式　$k^4(k+1)^4-(k-1)^4k^4=8k^7+8k^5$ に $k=1,2,\cdots,n$ を代入して，辺々加えると

$$n^4(n+1)^4=8\sum_{k=1}^{n}k^7+8\sum_{k=1}^{n}k^5$$

$$\therefore \sum_{k=1}^{n}k^7=\frac{1}{8}\left\{n^4(n+1)^4-8\times\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)\right\}=\frac{1}{24}n^2(n+1)^2\left\{3n^2(n+1)^2-2(2n^2+2n-1)\right\}$$

$$=\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)\quad ■$$

8.　2 つの恒等式　$k^5(k+1)^4-(k-1)^5k^4=9k^8-4k^7+14k^6-4k^5+k^4$，

$k^4(k+1)^5-(k-1)^4k^5=9k^8+4k^7+14k^6+4k^5+k^4$ を辺々加えると

$k^4(k+1)^4(2k+1)-(k-1)^4k^4(2k-1)=18k^8+28k^6+2k^4$

これに $k=1,2,\cdots,n$ を代入して，辺々加えると

$$n^4(n+1)^4(2n+1)=18\sum_{k=1}^{n}k^8+28\sum_{k=1}^{n}k^6+2\sum_{k=1}^{n}k^4$$

$$\therefore \sum_{k=1}^{n}k^8=\frac{1}{18}\left\{n^4(n+1)^4(2n+1)-28\times\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)-2\times\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)\right\}$$

$$=\frac{1}{270}n(n+1)(2n+1)\left\{15n^3(n+1)^3-10(3n^4+6n^3-3n+1)-(3n^2+3n-1)\right\}$$

$$=\frac{1}{270}n(n+1)(2n+1)\left\{15n^3(n+1)^3-10(3n^4+6n^3-3n+1)-(3n^2+3n-1)\right\}$$

$$=\frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)\quad ■$$

9.　恒等式　$k^5(k+1)^5-(k-1)^5k^5=10k^9+20k^7+2k^5$ に $k=1,2,\cdots,n$ を代入して，辺々加えると

$$n^5(n+1)^5=10\sum_{k=1}^{n}k^9+20\sum_{k=1}^{n}k^7+2\sum_{k=1}^{n}k^5$$

$$\therefore \sum_{k=1}^{n}k^9=\frac{1}{10}\left\{n^5(n+1)^5-20\times\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)-2\times\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)\right\}$$

$$=\frac{1}{60}n^2(n+1)^2\left\{6n^3(n+1)^3-5(3n^4+6n^3-n^2-4n+2)-(2n^2+2n-1)\right\}$$

$$=\frac{1}{20}n^2(n+1)^2(2n^6+6n^5+n^4-8n^3+n^2+6n-3)=\frac{1}{20}n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)\quad ■$$

10. 2つの恒等式　$k^6(k+1)^5-(k-1)^6k^5=11k^{10}-5k^9+30k^8-10k^7+7k^6-k^5$,

$k^5(k+1)^6-(k-1)^5k^6=11k^{10}+5k^9+30k^8+10k^7+7k^6+k^5$ を辺々加えると

$k^5(k+1)^5(2k+1)-(k-1)^5k^5(2k-1)=22k^{10}+60k^8+14k^6$

これに $k=1,2,\cdots,n$ を代入して，辺々加えると

$$n^5(n+1)^5(2n+1)=22\sum_{k=1}^{n}k^{10}+60\sum_{k=1}^{n}k^8+14\sum_{k=1}^{n}k^6$$

$$\therefore \sum_{k=1}^{n}k^{10}$$

$$=\frac{1}{22}\left\{n^5(n+1)^5(2n+1)-60\times\frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)-14\times\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)\right\}$$

$$=\frac{1}{66}n(n+1)(2n+1)\left\{3n^4(n+1)^4-2(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)-(3n^4+6n^3-3n+1)\right\}$$

$$=\frac{1}{66}n(n+1)(2n+1)(3n^8+12n^7+8n^6-18n^5-10n^4+24n^3+2n^2-15n+5)$$

$$=\frac{1}{66}n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)$$ ■

（参考）

$\sum_{k=1}^{n}k^{11}=\frac{1}{24}n^2(1+n)^2(10-20n-3n^2+26n^3-5n^4-16n^5+4n^6+8n^7+2n^8)$,

$\sum_{k=1}^{n}k^{12}=\frac{1}{2730}n(1+n)(1+2n)(-691+2073n-287n^2-3285n^3+1420n^4+2310n^5-1190n^6-1050n^7+$
$525n^8+525n^9+105n^{10})$,

$\sum_{k=1}^{n}k^{13}=\frac{1}{420}n^2(1+n)^2(-691+1382n+202n^2-1786n^3+367n^4+1052n^5-326n^6-400n^7+125n^8+$
$150n^9+30n^{10})$,

$\sum_{k=1}^{n}k^{14}=\frac{1}{90}n(1+n)(1+2n)(105-315n+44n^2+498n^3-217n^4-345n^5+182n^6+144n^7-81n^8-45n^9+$
$24n^{10}+18n^{11}+3n^{12})$,

$\sum_{k=1}^{n}k^{15}=\frac{1}{48}n^2(1+n)^2(420-840n-122n^2+1084n^3-226n^4-632n^5+203n^6+226n^7-83n^8-60n^9+$
$21n^{10}+18n^{11}+3n^{12})$,

$\sum_{k=1}^{n}k^{16}=\frac{1}{510}n(1+n)(1+2n)(-3617+10851n-1519n^2-17145n^3+7485n^4+11835n^5-6275n^6-4845n^7+$
$2775n^8+1365n^9-805n^{10}-315n^{11}+175n^{12}+105n^{13}+15n^{14})$,

$\sum_{k=1}^{n}k^{17}=\frac{1}{180}n^2(1+n)^2(-10851+21702n+3147n^2-27996n^3+5857n^4+16282n^5-5271n^6-5740n^7+$
$2165n^8+1410n^9-565n^{10}-280n^{11}+105n^{12}+70n^{13}+10n^{14})$,

$\sum_{k=1}^{n}k^{18}=\frac{1}{3990}n(1+n)(1+2n)(219335-658005n+92162n^2+1039524n^3-454036n^4-716940n^5+$
$380576n^6+292152n^7-167958n^8-80430n^9+48132n^{10}+16464n^{11}-9996n^{12}-2940n^{13}+1680n^{14}+$
$840n^{15}+105n^{16})$,

$$\sum_{k=1}^{n} k^{19} = \frac{1}{840} n^2(1+n)^2(438670 - 877340n - 127173n^2 + 1131686n^3 - 236959n^4 - 657768n^5 + 213337n^6 + 231094n^7 - 87665n^8 - 55764n^9 + 22835n^{10} + 10094n^{11} - 4263n^{12} - 1568n^{13} + 616n^{14} + 336n^{15} + 42n^{16}),$$

$$\sum_{k=1}^{n} k^{20} = \frac{1}{6930} n(1+n)(1+2n)(-3666831 + 11000493n - 1540967n^2 - 17378085n^3 + 7591150n^4 + 11982720n^5 - 6362660n^6 - 4877460n^7 + 2806470n^8 + 1335510n^9 - 801570n^{10} - 266310n^{11} + 163680n^{12} + 41580n^{13} - 25740n^{14} - 5940n^{15} + 3465n^{16} + 1485n^{17} + 165n^{18})$$

【参考文献】

[1]　初等数学第 84 号「最速求和法」（白坂　繁氏）（2018 年 9 月号）

[2]　$m = 11, 12, \cdots, 20$ のときは，Mathematica で計算

（2018/12/12　時岡）